

# CAN ONE HEAR FASTENING OF A ROD?

Akhtyamov A.M., Mouftakhov A.V.,  
Teicher M., Yamilova L.S.

(Bashkir State University, Ufa;  
Bar-Ilan University, Ramat-Gan, Israel)

**1. Introduction.** Rods are parts of various devices (see [1]–[5]). If it is impossible to observe the rod directly, the only source of information about possible defects of its fastening can be the natural frequencies of its flexural vibrations. The question arises whether one would be able to detect damage in rod fastening by the natural frequencies of its flexural vibrations. This paper gives and substantiates a positive answer to this question.

The problem in question belongs to the class of inverse problems and is a completely natural problem of identification of the boundary conditions.

Closely related formulations of the problem were proposed in [6, 7]. Contrary to this, in this paper it is not the form of the domain or size of an object which are sought for but the nature of fastening. The problem of determining a boundary condition has been considered in [8]. However, as data for finding the boundary conditions, we take not a set of natural frequencies, but not condensation and inversion (as in [8]).

Similarly formulated problems also occur in the spectral theory of differential operators, where it is required to determine the coefficients of a differential equation and the boundary conditions using a set of eigenvalues (for more details, see [9]–[14]). However, as data for finding the boundary conditions, we take one spectrum but not several spectra or other additional spectral data (for example, the spectral function, the Weyl function or the so-called weighting numbers) that were used in [9]–[13]. Moreover, the principal aim there was to determine the coefficients in the equation and not in boundary conditions. The aim of this paper is to determine the boundary conditions of the eigenvalue problem from its spectrum in the case of a known differential equation.

The problem of determining a boundary condition using a finite set of eigenvalues has been considered previously in [15]–[17]. In contrast to papers [15]–[16], we think it is necessary to determine not the type of fastening of plates, but the type of fastening of a rod. The problem

of finding of fastening of one of the end of a rod is considered in [17]. In contrast to papers [17] we think it is necessary to determine not the type of fastening of one of the ends of a rod, but the type of fastening of both ends of a rod.

**2. Formulation of the direct problem.** The problem of flexural oscillations of rod is reduced to following boundary problem [1, 2]

$$(\alpha u'')'' = \rho F \ddot{u}, \quad (1)$$

$$[(\alpha u'')' - c_1^- u]_{x=0} = 0, \quad [\alpha u'' - c_2^- u]_{x=0} = 0, \quad (2)$$

$$[(\alpha u'')' + c_1^+ u']_{x=l} = 0, \quad [\alpha u'' + c_2^+ u']_{x=l} = 0, \quad (3)$$

where  $\alpha$  is the flexural rigidity,  $l$  is the length,  $\rho$  is the density and  $F$  is the cross-section area of the rod,  $c_i^\pm$  ( $0 \leq c_i^\pm \leq \infty$ ) are rigidity coefficients of springs.

Let  $\alpha$ ,  $\rho$  and  $F$  be constant. By  $\lambda^2$  denote  $\rho F \omega^2 / \alpha$ .

For vibrations, we write

$$u(x, t) = y(x) \cos \omega t$$

and hence obtain the following eigenvalue problem for  $y(x)$ :

$$y^{(4)} = s^2 y, \quad (4)$$

$$U_1(y) = -a_1 y(0) + a_4 y'''(0) = 0, \quad U_2(y) = -a_2 y'(0) + a_3 y''(0) = 0, \quad (5)$$

$$U_3(y) = a_5 y(l) + a_8 y'''(l) = 0, \quad U_4(y) = a_6 y'(l) + a_7 y''(l) = 0, \quad (6)$$

where  $s^2 = \rho F \omega^2 / \alpha$ ,  $a_i \geq 0$  ( $i = 1, 2, \dots, 8$ ).

The coefficients  $a_i$  characterize conditions for fastening the rod (rigid clamping, free support, free edge, floating fixing, elastic fixing).

We note that the functions

$$\begin{aligned} y_1(x, s) &= (\cos \sqrt{s} x + \cosh \sqrt{s} x)/2, \\ y_2(x, s) &= (\sin \sqrt{s} x + \sinh \sqrt{s} x)/(2\sqrt{s}), \\ y_3(x, s) &= (-\cos \sqrt{s} x + \cosh \sqrt{s} x)/(2s), \\ y_4(x, s) &= (-\sin \sqrt{s} x + \sinh \sqrt{s} x)/(2\sqrt{s^3}), \end{aligned} \quad (7)$$

are linearly independent solutions of the equation (4) which satisfy the conditions

$$y_j^{(r-1)}(0, s) = \begin{cases} 0 & \text{when } j \neq r, \\ 1 & \text{when } j = r, \end{cases} \quad j, r = 1, 2, 3, 4. \quad (8)$$

The natural frequencies  $\omega_i$  ( $s_i$ ) are the corresponding positive eigenvalues of problem (4)–(6) (see [2, 5]). The non-zero eigenvalues of problem (4)–(6) are the roots of the determinant

$$\Delta(s) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) \end{vmatrix}$$

We shall now formulate the direct eigenvalue problem (4)–(6): it is required to find the unknown natural frequencies of the oscillations of the rod from  $a_i$  ( $i = 1, 2, \dots, 8$ ).

Thus, finding the natural frequencies  $\omega_i$  ( $i = 1, 2, \dots$ ) is equivalent to founding of the roots  $s_i$  ( $i = 1, 2, \dots$ ) of  $\Delta(s)$ .

Thus, knowing  $a_i$  ( $i = 1, 2, \dots, 8$ ) it is possible to find  $s_i$  ( $i = 1, 2, \dots$ ) by standard methods [2, 5]. The solution to this direct problem presents no difficulties. The question arises whether one would be able to do the reverse and find  $a_i$  ( $i = 1, 2, \dots, 8$ ) knowing  $s_i$  ( $i = 1, 2, \dots$ ). In a broader sense it may be stated as follows. Is it possible to derive unknown boundary conditions with a knowledge of  $s_i$ ? The answer to this question is given in the next section.

**3. Formulation of the inverse problem.** We shall denote the matrix, consisting of the coefficients  $a_j$  ( $j = 1, 2, \dots, 8$ ) of the forms  $U_i(y)$  ( $i = 1, 2, 3, 4$ ) by  $A$  and its minors by  $M_{ijkm}$  ( $i, j, m, k = 1, 2, \dots, 8$ ):

$$A = \begin{vmatrix} a_1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_7 & 0 \end{vmatrix}, \quad M_{ijkm} = \pm a_i a_j a_k a_m.$$

In terms of eigenvalue problem (4)–(6), the inverse problem constructed above should be formulated as follows: the coefficients  $a_j$  ( $j = 1, 2, \dots, 8$ ) of the forms  $U_i(y)$  ( $i = 1, 2, 3, 4$ ) of problem (4)–(6) are unknown, the rank of the matrix  $A$  to make up these coefficients is equal to four, the eigenvalues  $s_k$  of problem (4)–(6) are known and it is required to find the matrix  $A$  (the class of linear equivalent matrixes).

**4. The duality of the solution.** Together with problem (4)–(6), let us consider the following eigenvalue problem

$$y^{(4)} = s^2 y, \tag{9}$$

$$\widetilde{U}_1(y) = -\widetilde{a}_1 y(0) + \widetilde{a}_4 y'''(0) = 0, \quad \widetilde{U}_2(y) = -\widetilde{a}_2 y'(0) + \widetilde{a}_3 y''(0) = 0, \quad (10)$$

$$\widetilde{U}_3(y) = \widetilde{a}_5 y(l) + \widetilde{a}_8 y'''(l) = 0, \quad \widetilde{U}_4(y) = \widetilde{a}_6 y'(l) + \widetilde{a}_7 y''(l) = 0. \quad (11)$$

We denote the matrix composed of the coefficients  $\widetilde{a}_i$  of the forms  $\widetilde{U}_j(y)$  by  $\widetilde{A}$  and its minors by  $\widetilde{M}_{ijkm}$ :

$$\widetilde{A} = \begin{vmatrix} \widetilde{a}_1 & 0 & 0 & \widetilde{a}_4 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{a}_2 & \widetilde{a}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{a}_5 & 0 & 0 & \widetilde{a}_8 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{a}_6 & \widetilde{a}_7 & 0 \end{vmatrix}, \quad \widetilde{M}_{ijkm} = \pm \widetilde{a}_i \widetilde{a}_j \widetilde{a}_k \widetilde{a}_m.$$

Let  $\widetilde{A}^*$  be matrix

$$\widetilde{A}^* = \begin{vmatrix} \widetilde{a}_5 & 0 & 0 & \widetilde{a}_8 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{a}_6 & \widetilde{a}_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{a}_1 & 0 & 0 & \widetilde{a}_4 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{a}_2 & \widetilde{a}_3 & 0 \end{vmatrix}.$$

*Theorem 1 (on the duality of the solution of the inverse problem).* Suppose the following conditions are satisfied

$$\text{rank } A = \text{rank } \widetilde{A} = 4. \quad (12)$$

If the eigenvalues  $\{s_i\}$  of problem (4)–(6) and the eigenvalues  $\{\widetilde{s}_i\}$  of problem (9)–(11) are identical, with account taken for their multiplicities, the classes of linearly equivalent matrixes  $A$  and  $\widetilde{A}$  or  $A$  and  $\widetilde{A}^*$  are also identical.

*Proof.* The eigenvalues of problem (4)–(6) are the roots of the determinant

$$\begin{aligned} \Delta(s) = & -M_{1256} [f^-(s)/s^2] + (M_{2457} + M_{1368}) [f^-(s)] + M_{3478} [s^2 f^-(s)] \\ & + (M_{1278} + M_{3456} - M_{2457} - M_{1368}) [f^+(s)] + M_{1357} [z(s)/s] - M_{2468} [s z(s)] \\ & + (M_{2456} + M_{1268}) [g^-(s)/\sqrt{s}] - (M_{3468} + M_{2478}) [\sqrt{s^3} g^-(s)] + \\ & + (M_{1356} + M_{1257}) [g^+(s)/\sqrt{s^3}] - (M_{3457} + M_{1378}) [\sqrt{s} g^+(s)], \end{aligned} \quad (13)$$

where

$$f^\pm(s) = (1 \pm \cos \sqrt{s} \cosh \sqrt{s})/2, \quad z(s) = (\sin \sqrt{s} \sinh \sqrt{s})/2,$$

$$g^\pm(s) = (-\sin \sqrt{s} \cosh \sqrt{s} \pm \cos \sqrt{s} \sinh \sqrt{s})/2.$$

Note that the functions in square brackets are linearly independent. Since  $\Delta(s) \neq 0$ ,  $\tilde{\Delta}(s) \neq 0$  are entire functions in  $s$  of order  $1/2$ , it follows from Hadamard's factorization theorem (see [18]) that determinants  $\Delta(s)$  and  $\tilde{\Delta}(s)$  are connected by the relation

$$\Delta(s) \equiv C \tilde{\Delta}(s),$$

where  $k$  is a certain integer and  $C$  is a certain non-zero constant. From this and we obtain the equalities

$$M_{1256} = K \tilde{M}_{1256}, \quad (14)$$

$$M_{1357} = K \tilde{M}_{1357}, \quad (15)$$

$$M_{2468} = K \tilde{M}_{2468}, \quad (16)$$

$$M_{3478} = K \tilde{M}_{3478}, \quad (17)$$

$$M_{1257} + M_{1356} = K (\tilde{M}_{1356} + \tilde{M}_{1257}), \quad (18)$$

$$M_{1268} + M_{2456} = K (\tilde{M}_{1268} + \tilde{M}_{2456}), \quad (19)$$

$$M_{1378} + M_{3457} = K (\tilde{M}_{1378} + \tilde{M}_{3457}), \quad (20)$$

$$M_{2478} + M_{3468} = K (\tilde{M}_{2478} + \tilde{M}_{3468}), \quad (21)$$

$$M_{1278} + M_{3456} = K (\tilde{M}_{1278} + \tilde{M}_{3456}), \quad (22)$$

$$M_{1368} + M_{2457} = K (\tilde{M}_{1368} + \tilde{M}_{2457}). \quad (23)$$

Hence,

$$a_1 = \tilde{a}_1, \quad a_2 = \tilde{a}_2, \quad a_3 = \tilde{a}_3, \quad a_4 = \tilde{a}_4,$$

$$a_5 = \tilde{a}_5, \quad a_6 = \tilde{a}_6, \quad a_7 = \tilde{a}_7, \quad a_8 = \tilde{a}_8,$$

or

$$a_1 = \tilde{a}_5, \quad a_2 = \tilde{a}_6, \quad a_3 = \tilde{a}_7, \quad a_4 = \tilde{a}_8,$$

$$a_5 = \tilde{a}_1, \quad a_6 = \tilde{a}_4, \quad a_7 = \tilde{a}_3, \quad a_8 = \tilde{a}_4.$$

These equations can be proved by standard method of algebra. Let us consider only one case  $M_{3478} \neq 0$  for example. If  $M_{3478} \neq 0$ ,  $a_3 \neq 0$ ,  $a_4 \neq 0$ ,  $a_7 \neq 0$ ,  $a_8 \neq 0$ ,  $\tilde{a}_3 \neq 0$ ,  $\tilde{a}_4 \neq 0$ ,  $\tilde{a}_7 \neq 0$ ,  $\tilde{a}_8 \neq 0$  and so matrixes  $A$  and  $\tilde{A}$  have the forms

$$A = \begin{vmatrix} a_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 1 & 0 \end{vmatrix}, \quad \tilde{A} = \begin{vmatrix} \tilde{a}_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \tilde{a}_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{a}_5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \tilde{a}_6 & 1 & 0 \end{vmatrix}.$$

From this and from (17) we obtain  $K = 1$ .

It follows from (15), (20) that

$$a_1 a_5 = \tilde{a}_1 \tilde{a}_5, \quad a_1 + a_5 = \tilde{a}_1 + \tilde{a}_5.$$

Therefore by Vieta's theorem, we have

$$a_1 = \tilde{a}_1, \quad a_5 = \tilde{a}_5 \quad \text{or} \quad a_1 = \tilde{a}_5, \quad a_5 = \tilde{a}_1. \quad (24)$$

Similarly, equations (16), (21) imply

$$a_2 = \tilde{a}_2, \quad a_6 = \tilde{a}_6, \quad \text{or} \quad a_2 = \tilde{a}_6, \quad a_6 = \tilde{a}_2. \quad (25)$$

These equations are equivalent to four cases:

Case 1:  $a_1 = \tilde{a}_1, \quad a_5 = \tilde{a}_5, \quad a_2 = \tilde{a}_2, \quad a_6 = \tilde{a}_6$ .

Case 2:  $a_1 = \tilde{a}_5, \quad a_5 = \tilde{a}_1, \quad a_2 = \tilde{a}_6, \quad a_6 = \tilde{a}_2$ .

Case 3:  $a_1 = \tilde{a}_1, \quad a_5 = \tilde{a}_5, \quad a_2 = \tilde{a}_6, \quad a_6 = \tilde{a}_2$ .

Case 4:  $a_1 = \tilde{a}_5, \quad a_5 = \tilde{a}_1, \quad a_2 = \tilde{a}_2, \quad a_6 = \tilde{a}_6$ .

Cases 3 and 4 are special cases of 1 and 2. Indeed, in case 3 it follows from (22) or (23) that

$$(a_1 - a_5)(a_2 - a_6) = 0. \quad (26)$$

Hence  $a_1 - a_5 = 0$  or  $a_2 - a_6 = 0$ . If  $a_1 - a_5 = 0$ , then case 3 is special case 2:  $\tilde{a}_1 = a_1 = a_5 = \tilde{a}_5$ . If  $a_2 - a_6 = 0$ , then case 3 is special case 1:  $\tilde{a}_6 = a_2 = a_6 = \tilde{a}_6$ .

In the same way, we can show that the case 4 is special case of cases 1 and 2. We have proved that in the case  $M_{3478} \neq 0$  it is possible only two cases: 1 and 2. In a similar manner we can prove the analogous results for other cases. This contradiction proves the theorem.

**5. Exact solution.** It has been shown above that the problem of finding the matrix  $A$  (in the sense that class of linearly equivalent matrixes of  $A$ ) from the natural frequencies of flexural oscillations of a rod has a duality solution. The next question is how this solution can be constructed.

This section deals with solving this problem and constructing exact solution by the first 9 natural frequencies  $\omega_i$ .

Suppose  $s_1, s_2, \dots, s_9$  are the values corresponding to the first nine natural frequencies  $\omega_i$ . We substitute the values  $s_1, s_2, \dots, s_9$  into (13)

and obtain a system of nine homogeneous algebraic equations

$$\begin{aligned}
& x_1 [f^-(s_i)/s_i^2] + x_2 [f^-(s_i)] + x_3 [s_i^2 f^-(s_i)] \\
& + x_4 [f^+(s_i)] + x_5 [z(s_i)/s_i] + x_6 [s_i z(s_i)] \\
& + x_7 [g^-(s_i)/\sqrt{s_i}] + x_8 [\sqrt{s_i^3} g^-(s_i)] + \\
& + x_9 g^+(s_i)/\sqrt{s_i^3} + x_{10} [\sqrt{s_i} g^+(s_i)] = 0, \quad i = 1, 2, \dots, 9
\end{aligned} \tag{27}$$

in the ten unknowns

$$\begin{aligned}
x_1 &= -M_{1256}, & x_2 &= M_{2457} + M_{1368}, & x_3 &= M_{3478}, \\
x_4 &= M_{1278} + M_{3456} - M_{2457} - M_{1368}, & x_5 &= M_{1357}, \\
x_6 &= -M_{2468}, & x_7 &= M_{2456} + M_{1268}, & x_8 &= -(M_{3468} + M_{2478}), \\
x_9 &= M_{1356} + M_{1257}, & x_{10} &= -(M_{3457} + M_{1378}).
\end{aligned} \tag{28}$$

The resulting set of equations (27) has an infinite number of solutions. If the resulting set has a rank of 9, the unknown  $x_i$  can be found in accurate to a coefficient. The unknown matrix  $A$  is found from (31) by direct calculations as in proof of theorem 1.

These reasons prove

*Theorem 2 (on the duality of the solution of the inverse problem).* If the matrix of system (27) has a rank of 9, the solution of the inverse problem of the reconstruction boundary conditions (10), (11) is duality.

*Remark.* Theorem 2 is stronger than theorem 1. Theorem 2 use only 9 natural frequencies for the reconstruction of boundary conditions and not all natural frequencies as in theorem 1. But theorem 1 is proves duality of the solution in the common case (if the set (27) has not a rank of 9).

Continuity of the solution of the inverse problem with respect to  $s_i$  is proved as in [16]. This shows that small perturbations of eigenvalues  $s_i$  ( $i = 1, 2, 3$ ) lead to small perturbations of the boundary conditions. It follows from this and theorem 1 that the inverse problem is well posed, since its solution exists, is unique and continuous with respect to  $s_i$  ( $i = 1, 2, 3$ ).

Computer calculations confirm the stability of the solution of the inverse problem. The order of error often hardly different from the error in the closeness of values of  $\tilde{s}_k$  and  $s_i$  and only in some cases it can be deteriorated by four orders of magnitude. So the measurement accuracy of instruments to measure natural frequencies must exceed accuracy to measure boundary conditions by four orders of magnitude.

**3. Numerical results.** We use dimensionless variables in the numerical examples.

*Example 1 (rigid clamping – free support).* Suppose

$$\begin{aligned} s_1 &= 15.4182057169801, & s_2 &= 49.9648620318002, \\ s_3 &= 104.247696458861, & s_4 &= 178.269729494609, \\ s_5 &= 272.030971305025, & s_6 &= 385.531421917553, \\ s_7 &= 518.771081332259, & s_8 &= 671.749949549144, \\ s_9 &= 844.468026568208 \end{aligned}$$

correspond to the first 9 natural frequencies  $\omega_i$  determined using instruments for measuring the natural frequencies, then the solution of set (27) has the form

$$\begin{aligned} x_1 &= -M_{1256} \approx 0 \cdot C, & x_2 &= M_{2457} + M_{1368} \approx 0 \cdot C, \\ x_3 &= M_{3478} \approx 0 \cdot C, & x_4 &= M_{1278} + M_{3456} - M_{2457} - M_{1368} \approx 0 \cdot C, \\ x_5 &= M_{1357} \approx 0 \cdot C, & x_6 &= -M_{2468} \approx 0 \cdot C, \\ x_7 &= -M_{2456} + M_{1268} \approx 0 \cdot C, & x_8 &= -(M_{3468} + M_{2478}) \approx 0 \cdot C, \\ x_9 &= M_{1356} + M_{1257} \approx C, & x_{10} &= -(M_{3457} + M_{1378}) \approx 0 \cdot C \end{aligned} \tag{29}$$

with an accuracy of  $10^{-9}$ .

Substituting 1 for  $C$  in (29), we get  $M_{1356} + M_{1257} = 1$ . This means that  $M_{1356} \neq 0$  or  $M_{1257} \neq 0$ .

Suppose  $M_{1356} \neq 0$ . Then  $a_1 a_3 a_5 a_6 \neq 0$ . So

$$A = \left\| \begin{array}{cccccccc} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_7 & 0 \end{array} \right\|$$

(in the sense that class of linearly equivalent matrixes of  $A$ ).

Using  $M_{1256} = 0$ ,  $M_{1357} = 0$  and  $M_{2457} + M_{1368} = 0$ , we have  $a_2 = 0$ ,  $a_7 = 0$  and  $a_8 = 0$ . From this and  $M_{1278} + M_{3456} - M_{2457} - M_{1368} = 0$ , we get  $a_4 = 0$ .

Thus we have

$$A = \left\| \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right\|$$



Suppose now  $M_{1257} \neq 0$ . Then  $a_1 a_2 a_5 a_7 \neq 0$ . So

$$A = \begin{vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 1 & 0 \end{vmatrix}$$

Using  $M_{1256} = 0$  and  $M_{2457} + M_{1368} = 0$ , we get  $a_6 = 0$  and  $a_4 = 0$ .

If we combine this with equations  $M_{1357} = 0$  and  $M_{1278} + M_{3456} - M_{2457} - M_{1368} = 0$ , we get  $a_3 = 0$  and  $a_8 = 0$ .

Thus we have

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

So boundary conditions are or

$$U_1(y) = y(0) = 0, \quad U_2(y) = y''(0) = 0,$$

$$U_3(y) = y(l) = 0, \quad U_4(y) = y'(l) = 0,$$

either

$$U_1(y) = y(0) = 0, \quad U_2(y) = y'(0) = 0,$$

$$U_3(y) = y(l) = 0, \quad U_4(y) = y''(l) = 0.$$

Note that the numbers  $s_i$  presented above are the same as the first nine exact values corresponding to fastening of rigid clamping and free support. This means that the unknown rod fastening inaccessible to direct observation has been correctly determined.

*Example 2 (rigid clamping – elastic fixing).* If

$$\begin{aligned} s_1 &= 5.60163863016235, & s_2 &= 22.4984332740862, \\ s_3 &= 61.8604321649037, & s_4 &= 120.984868139371, \\ s_5 &= 199.909638169628, & s_6 &= 298.589053349029, \\ s_7 &= 417.014779762035, & s_8 &= 555.183266366176, \\ s_9 &= 713.092945010199 \end{aligned}$$

correspond to the first 9 natural frequencies  $\omega_i$  determined using instruments for measuring the natural frequencies, then the solution of set

(27) has the form

$$\begin{aligned}
x_1 &= -M_{1256} \approx 0, & x_2 &= M_{2457} + M_{1368} \approx 0, \\
x_3 &= M_{3478} \approx 0, & x_4 &= M_{1278} + M_{3456} - M_{2457} - M_{1368} \approx -C, \\
x_5 &= M_{1357} \approx 0, & x_6 &= -M_{2468} \approx 0, \\
x_7 &= -M_{2456} + M_{1268} \approx 0, & x_8 &= -(M_{3468} + M_{2478}) \approx 0, \\
x_9 &= M_{1356} + M_{1257} \approx 5C, & x_{10} &= -(M_{3457} + M_{1378}) \approx 0.
\end{aligned} \tag{30}$$

Suppose  $C = 1$ ; then  $M_{1356} + M_{1257} \neq 0$ . This means that  $M_{1356} \neq 0$  or  $M_{1257} \neq 0$ .

Suppose  $M_{1356} \neq 0$ . Then  $a_1 a_3 a_5 a_6 \neq 0$ . So

$$A = \begin{vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_7 & 0 \end{vmatrix}$$

(in the sense that class of linearly equivalent matrixes of  $A$ ).

If we combine this with equations  $M_{1256} = 0$ ,  $M_{1356} + M_{1257} = 5C$  we get  $a_2 = 0$ ,  $1 = 5C$ . Using  $M_{1357} = 0$  and  $M_{2457} + M_{1368} = 0$ , we have  $a_7 = 0$  and  $a_8 = 0$ . From this and  $M_{1278} + M_{3456} - M_{2457} - M_{1368} = -C$ , we get  $-a_4 = -1/5$ .

Thus we have

$$A = \begin{vmatrix} 1 & 0 & 0 & 1/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Suppose now  $M_{1257} \neq 0$ . Then  $a_1 a_2 a_5 a_7 \neq 0$ . So

$$A = \begin{vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 1 & 0 \end{vmatrix}$$

Using  $M_{1256} = 0$  and  $M_{2457} + M_{1368} = 0$ , we get  $a_6 = 0$  and  $a_4 = 0$ .

If we combine this with equations  $M_{1357} = 0$ ,  $M_{1356} + M_{1257} = 5C$ ,  $M_{1278} + M_{3456} - M_{2457} - M_{1368} = -C$  and  $M_{2457} + M_{1368} = 0$  we get  $a_3 = 0$ ,  $1 = 5C$ , and  $-a_8 = -1/5$ .

Thus we have

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

So boundary conditions are or

$$U_1(y) = -5y(0) + y'''(0) = 0, \quad U_2(y) = y''(0) = 0,$$

$$U_3(y) = y(l) = 0, \quad U_4(y) = y'(l) = 0,$$

either

$$U_1(y) = y(0) = 0, \quad U_2(y) = y'(0) = 0,$$

$$U_3(y) = 5y(l) + y'''(l) = 0, \quad U_4(y) = y''(l) = 0.$$

Note that the numbers  $s_i$  presented above are the same as the first nine exact values corresponding to fastening of rigid clamping and elastic fixing with the relative stiffness factor of 5. This means that the unknown rod fastening inaccessible to direct observation has been correctly determined.

*Example 3. (elastic fixing — elastic fixing)* If

$$\begin{aligned} s_1 &= 0.383848559322840, & s_2 &= 11.9180148367849, \\ s_3 &= 53.4326824121208, & s_4 &= 114.157790468867, \\ s_5 &= 193.836586296759, & s_6 &= 292.955617117120, \\ s_7 &= 411.667770695782, & s_8 &= 550.037492353361, \\ s_9 &= 708.096219400352 \end{aligned}$$

correspond to the first 9 natural frequencies  $\omega_i$  determined using instruments for measuring the natural frequencies with an accuracy of  $10^{-15}$ , then the solution of set (27) has the form

$$\begin{aligned} x_1 &= -M_{1256} \approx -24C, & x_2 &= M_{2457} + M_{1368} \approx -10C, \\ x_3 &= M_{3478} \approx C, & x_4 &= M_{1278} + M_{3456} - M_{2457} - M_{1368} \approx 9C, \\ x_5 &= M_{1357} \approx 3C, & x_6 &= -M_{2468} \approx -8C, \\ x_7 &= -M_{2456} + M_{1268} \approx -32C, & x_8 &= -(M_{3468} + M_{2478}) \approx -6C, \\ x_9 &= M_{1356} + M_{1257} \approx 18C, & x_{10} &= -(M_{3457} + M_{1378}) \approx 4C. \end{aligned} \tag{31}$$

Suppose  $C = 1$ ; then  $M_{3478} = a_3 a_4 a_7 a_8 \neq 0$ . So

$$A = \begin{vmatrix} a_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 1 & 0 \end{vmatrix}$$

(in the sense that class of linearly equivalent matrixes of  $A$ ).

From this and (31), we obtain or  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_5 = 3$ ,  $a_6 = 4$  either  $a_1 = 3$ ,  $a_2 = 4$ ,  $a_5 = 1$ ,  $a_6 = 2$ . So boundary conditions are or

$$U_1(y) = -y(0) + y'''(0) = 0, \quad U_2(y) = -2y'(0) + y''(0) = 0,$$

$$U_3(y) = 3y(l) + y'''(l) = 0, \quad U_4(y) = 4y'(l) + y''(l) = 0,$$

either

$$U_1(y) = -3y(0) + y'''(0) = 0, \quad U_2(y) = -4y'(0) + y''(0) = 0,$$

$$U_3(y) = y(l) + y'''(l) = 0, \quad U_4(y) = 2y'(l) + y''(l) = 0.$$

Note that the numbers  $s_i$  presented above are the same as the first nine exact values corresponding to elastic fixing with the relative stiffness factors of 1, 2, 3, 4. This means that the unknown rod fastening inaccessible to direct observation has been correctly determined.

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